A PRIORI ESTIMATES OF STANDARD ERRORS OF LEVELING DATA

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BIOGRAPHICAL SKETCH

David B. Zilkoski received a B.S. degree in Forest Engineering from the College of Environmental Science and Forestry, Syracuse, New York, in 1974, and an M.S. degree in Geodetic Science from the Ohio State University in 1979. He has been employed by the National Geodetic Survey (NGS) since 1974. From 1974 to 1981, as a member of the Horizontal Network Branch, he participated in the new adjustment of the North American Datum of 1983. His present position is Geodesist, Vertical Network Branch, and Project Manager, New Adjustment of the North American Vertical Datum of 1988.

Mr. Zilkoski is a member of the American Congress on Surveying and Mapping (ACSM) and is an instructor for the NGS Vertical Control Workshop. He is also a member of the American Geophysical Union and President of International Association of Geodesy Special Study Group 1.102, "Vertical Reference Systems."

ABSTRACT

Differential leveling observations are relative height differences measured between bench marks. Periodic adjustments are necessary as leveling observations are incorporated into the existing network. When combining the same order, class, and age of leveling data in relatively small network adjustments, the relative weighting scheme is usually not considered to be significant. In larger vertical control network adjustments, where many different orders and classes, as well as different ages of leveling data are combined, the relative weighting scheme is extremely important. If the a priori estimates of standard errors of leveling observations are incorrect, the observations will not receive the appropriate corrections and the adjusted heights will be incorrectly estimated. Also, post-adjustment error analysis can produce incorrect uncertainty values for adjusted results. An analysis of estimates of standard errors of leveling lines indicates that leveling data obtained by the National Geodetic Survey after 1978 are significantly more precise than data obtained in 1978 and earlier.

INTRODUCTION

Differential leveling observations are relative height differences measured between bench marks. The bench marks' heights and leveling observations are related through the following linear model:

$$H_j - H_i = L_b$$

where

H_i = height of bench mark i,

H_j = height of bench mark j, and

L_b = observed height difference between bench mark i and bench mark j.

In the North American Vertical Datum of 1988 (NAVD 88) readjustment project, as in most leveling networks, there are more observations than unknowns, i.e., the number of observed height differences exceeds the number of unknown bench mark heights. This redundancy determines the degrees of freedom of an adjustment, i.e., the degrees of freedom equals the number of observations minus the number of unknown parameters. Hence, different adjusted values of bench mark heights can be obtained by using different combinations of leveling data.

For the NAVD 88 project, the classical least squares method of observation equations is being be used to perform the adjustment of the leveling data. The mathematical model of the method of observation equations can be represented by the following equation:

$$L_a = F(X_a),$$

where L_a is a set of adjusted observations (e.g., leveling height differences), X_a is a set of parameters (e.g., bench mark heights), and F is a function which relates the observations to the parameters.

The set of observation equations can be represented as

$$V + L = AX,$$

where

V = vector of residuals (discrepancies),
A = design matrix,
X = vector of parameters, and
L = vector of observations.

It can be shown that when the least squares condition of minimum sum of the weighted residuals squared is fulfilled, the normal equations will be

$$NX + U = 0,$$

where

$N = A^T PA$,

$U = A^T PL$,

$P = k(\text{Var-Cov})^{-1}$,

Var-Cov = symmetric, positive definite, variance-covariance matrix of observations.

The least squares estimate of X is obtained from

$$X = -N^{-1}U.$$

In order to solve for X, the matrix N must be of full rank. In other words, the rank of N must be equal to the number of unknowns. In a leveling network adjustment consisting only of measured height differences, N will not be of full rank, but will actually be equal to the number of unknowns minus one. Therefore, at least one parameter will have to be weighted when using the method of weighted parameters. By weighting one parameter, fixing it to its a priori estimate, N can be inverted and the solution of X obtained. This is called a minimum constraint least squares adjustment, or "free" adjustment.

The observation equations for differential leveling observations between station i and station j consist of the following:

$$V_k = H_j - H_i - L_{ij},$$

where

V_k = residual for observation k,

H_i = height of bench mark i,

H_j = height of bench mark j, and

L_{ij} = observed height difference from station i to station j.

The observed height differences are assumed to be uncorrelated; hence all off-diagonal terms of the variance-covariance matrix of the observations are equal to zero. The nonzero, diagonal terms are not easy to determine. When combining the same order, class, and age of leveling data in relatively small network adjustments, the relative weighting scheme is usually not considered to be significant. In larger vertical control network adjustments, where many different orders and classes, as well as different ages of leveling data are combined, having a correct relative weighting scheme is extremely important. The a priori estimates of standard errors for all orders and classes of leveling data are assumed to be known. The weight of a leveling observation (p_i) is determined using the formula 1/(variance of the observation i), where the variance of the observation i is equal to: the a priori standard error squared times the distance leveled, in kilometers, divided by the number of runnings.

If the a priori estimates of standard errors of the leveling data are incorrect, the observations will not receive their appropriate corrections and the adjusted heights will be incorrectly estimated. In addition, observations may be incorrectly flagged as data outliers and removed from the analysis. The following basic assumptions are made when performing least squares adjustments:

(1) All data outliers have been removed from the data.
(2) The mathematical model is correct.
(3) Correct relative and absolute weights have been imposed.

All systematic errors must be resolved when evaluating the mathematical model. If one or more of these assumptions are not valid, the heights obtained from the adjustment may be distorted.

ESTIMATION OF STANDARD ERRORS

The a priori standard errors of 1 km of single-run leveling for first- and second-order leveling used by the National Geodetic Survey (NGS) are listed below:

first-order, class 0 = 0.7 mm,
first-order, class I = 1.1 mm,
first-order, class II = 1.4 mm,

second-order, class I = 2.1 mm,
second-order, class II = 2.8 mm, and
second-order, class 0 = 3.0 mm.

The estimates of standard errors listed above were empirically determined in the late 1970s using a limited amount of data which were available in computer-readable form at the time the analysis was performed.

NGS' archival leveling data were processed and loaded into the NGS Integrated Data Base (NGSIDB) during the early 1980s. There are approximately 17,000 leveling lines in NGSIDB. In preparation for NAVD 88, a "standard error of 1 kilometer of single-run leveling" statistic was computed for each leveling line and loaded into NGSIDB. Approximately 14,000 leveling lines were used in this study. The 3,000 leveling lines not used consisted of Canadian leveling lines, and U.S. third-order data and single-run and second-order, class 0 area work which did not contain any double-run sections.

The formula given below was used to compute the standard error statistic:

$$\sigma_s = \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\sum_{j=1}^{m} [|\overline{\Delta h_i}| - |\Delta h_{ij}|]^2}{S_i(m-1)} \right] \right\}^{1/2}$$

where

σ_s = standard error of one km of single-run leveling,

n = number of sections with 2 or more non-rejected runnings,

m = number of non-rejected runnings in a section,

$\overline{\Delta h_i}$ = mean elevation difference for the i^{th} section,

Δh_{ij} = the j^{th} non-rejected elevation difference for section i, and

S_i = the length of section i in kilometers.

The leveling data were divided into six groups defined by major changes in leveling procedures and/or equipment as indicated below:

(1) Before 1902 - Use of "Vienna" or "Stampfer" type instruments and "paraffin" soaked wooden rods.

(2) Between 1902 and 1916 - New leveling instrument (Fischer level).

(3) Between 1917 and 1962 - New leveling rod (Invar rod).

(4) Between 1963 and 1970 - New type of instrument ("parallel-plate micrometer" instruments), new type of leveling rod (Kern double-scale Invar rods), and modified leveling procedures (reduced sight lengths and maximum differences in length of forward and backward sights at each setup were reduced).

(5) Between 1971 and 1978 - New leveling instrument (Zeiss Ni1 compensator instrument).

(6) After 1978 - New leveling instrument (Ni002 reversible compensator instrument), modified procedures (double-simultaneous, single-run leveling), introduction of automated recording system and low- and high-scale checks, and use of "motorized leveling" system.

DISCUSSION OF STANDARD ERROR ESTIMATES

The standard errors were plotted against the year the leveling lines was observed. Figure 1 depicts the standard error versus the year observed for first-order, class II leveling data. Figure 1 shows the obvious improvement in standard error estimates after 1970 for first-order, class II leveling data.

The weight of each line used in the weighted mean standard error estimate was computed using the formula: (number of runnings minus number of sections)/(number of sections). Thus, a leveling line that is double-run would get a weight of one. Table 1 gives the mean standard error estimate and weighted mean standard error estimate for all order and classes within each group for data in NGSIDB. Table 1 indicates that there is an improvement in standard errors for all data after 1970. For example, the standard error of first-order, class II leveling data for group 4 is 1.90, for group 5 it is 1.26, and for group 6 it is 1.01. This is a significant improvement in precision of leveling data. Similarly, the standard error of second-order, class I leveling data for group 5 is 1.28 and for group 6 it is 1.04. This implies that new procedures and/or instrumentation have improved the estimated precision of leveling data.

It should also be noted that the standard error estimates for second-order, class I leveling data are very similar to first-order, class II estimates for groups 5 and 6. For example, the standard error estimate for group 5 is 1.26 and the estimate for second-order, class I for group 5 is 1.28. This is probably because the specifications and procedures for performing first-order, class II and second-order, class I leveling are very similar. The only real differences in the procedures and specifications between the two orders and classes are the section and loop misclosure tolerances. Therefore, it is not surprising to find that the estimates of the standard errors for these two orders and classes are the same when a large sample of data is used.

In table 1, the statistics computed using all data are larger than the statistics computed using data which were obtained after 1978. What is also interesting to note is that the estimate for second-order, class I for all years is significantly less than the first-order, class II estimate for all years. This is because the standard errors are so large for first-order, class II leveling for groups 3 and 4 which contain over three-quarters of the total data for first-order, class II. This is a reason why the mean standard error estimated using all data should not be used for a priori estimates of standard errors. For example, the standard error estimated for first-order, class II leveling data using all years is 1.76 while the estimate is only 1.01 using data obtained after 1978.

The spread of the estimates of standard errors for first-order, class II appears to be large. The standard errors range from almost zero to 8.5. (See Figure 1). Figure 2 is a plot of the standard errors for first-order, class II leveling lines that were at least 50 percent double-run, i.e., the weight is equal to or greater than 0.5, against the number of bench marks on the leveling line. Notice that the large spread of standard errors appears to be for leveling lines containing less than 50 bench marks. This may indicate that for longer leveling lines, the computed statistic may be averaging out larger errors.

Table 2 gives the mean standard error estimates and weighted mean standard error estimate for all order and classes for each group using leveling lines that contain at least 50 bench marks and were at least 50 percent double-run. The standard errors increase in almost all cases, except for first-order, class I and first-order, class II leveling data. Although it should be noted that the sample size decreases significantly in all order and classes except for those two orders and classes that the standard errors values did not increase. Once again, this may indicate that the standard error statistic computed for longer leveling lines may be averaging out larger errors than the statistic implies. In another computation, the standard errors were also plotted against latitude and longitude to examine if the standard errors were location dependent. These plots did not show any apparent correlations.

CONCLUSION

Leveling data in NGS' Integrated Data Base that were obtained after 1978 appear to be significantly more precise than prior data. In addition, there tends to be an improvement in precision of all leveling data after an equipment and/or procedural change was imposed, indicating that the changes improved the precision of leveling data. The results of this study will be used to determine the standard errors of leveling data used in the NAVD 88. Special adjustments will be performed using preliminary standard errors values to assist in evaluating the results.

* * *

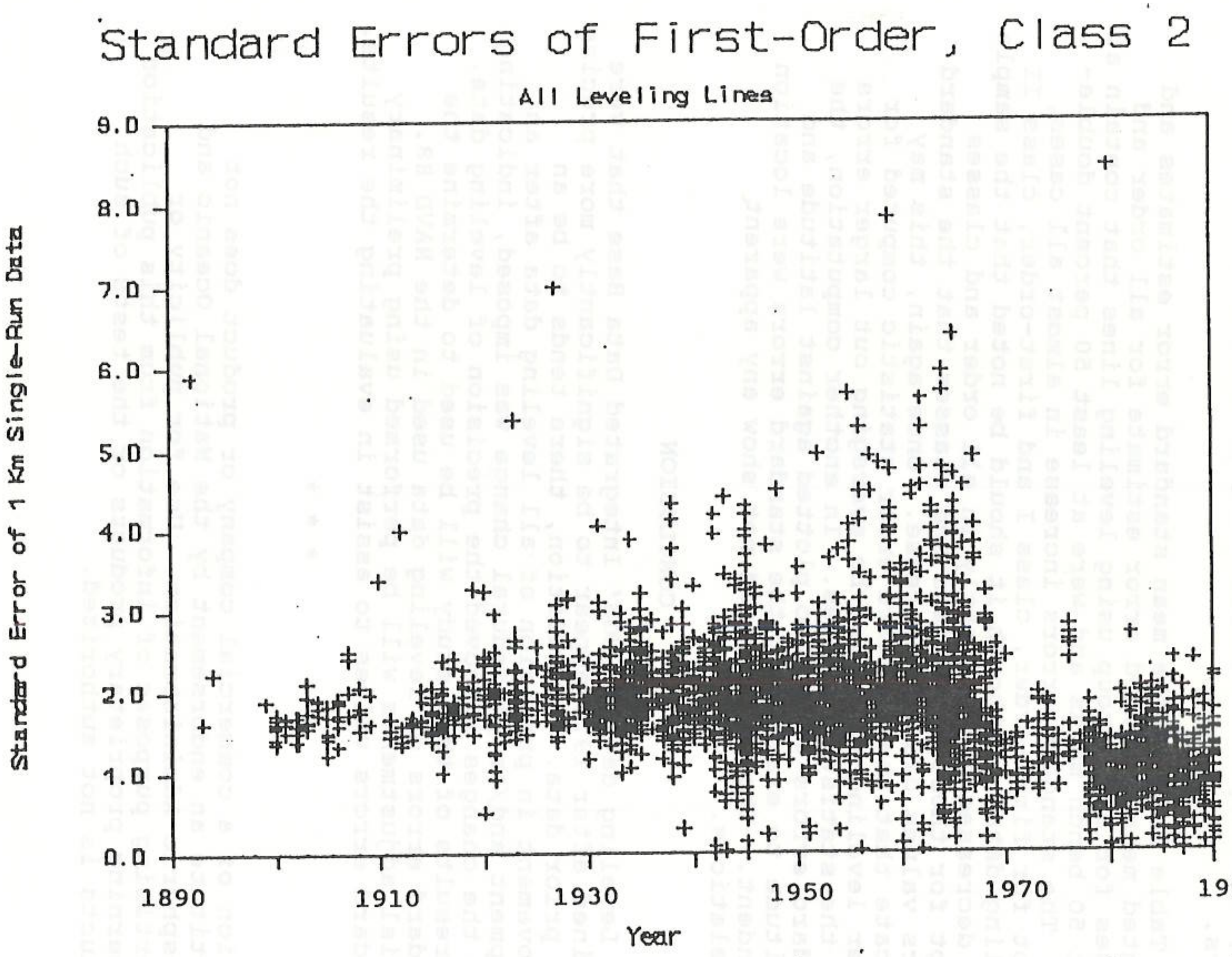

Figure 1. Plot of first-order, class II standard errors versus year observed

Table 1. Estimates of standard errors of leveling data in NGSIDB; all leveling lines were used to compute the statistics

Group	Order/Class					
	1/0	1/1	1/2	2/1	2/2	2/0
All Years	0.63	1.12	1.76	1.11	1.74	2.42
	0.63	1.16	1.93	1.16	2.00	2.57
	77	655	4,323	709	334	7,959
Before 1902	--	--	2.05	--	--	--
	--	--	1.99	--	--	--
	--	--	11	--	--	--
Between 1902	--	--	1.73	--	--	1.94
and 1916	--	--	1.78	--	--	1.94
	--	--	98	--	--	1
Between 1917	--	1.93	1.98	1.16	2.24	2.42
and 1962	--	1.92	2.02	1.75	2.24	2.58
	--	21	2613	8	1	6632
Between 1963	0.83	1.19	1.90	2.14	2.57	2.47
and 1970	0.86	1.23	1.95	2.14	2.59	2.62
	10	250	668	1	2	1100
Between 1971	0.83	1.06	1.26	1.28	1.77	1.97
and 1978	0.83	1.08	1.39	1.33	2.10	2.17
	31	329	109	206	192	226
After 1978	0.41	0.84	1.01	1.04	1.69	--
	0.41	0.83	1.03	1.07	1.90	--
	36	55	824	494	139	--

Weight - (No. Runs - No. Sections)/(No. Sections)
Group - Leveling lines which fall into age group were used in estimating the statistics.
1/0 - First-Order, Class 0
1/1 - First-Order, Class I
1/2 - First-Order, Class II
2/1 - Second-Order, Class I
2/2 - Second-Order, Class II
2/0 - Second-Order, Class 0

x.xx - Mean standard error estimate
y.yy - Weighted mean standard error estimate
nn - Number of leveling lines used in estimating statistic.

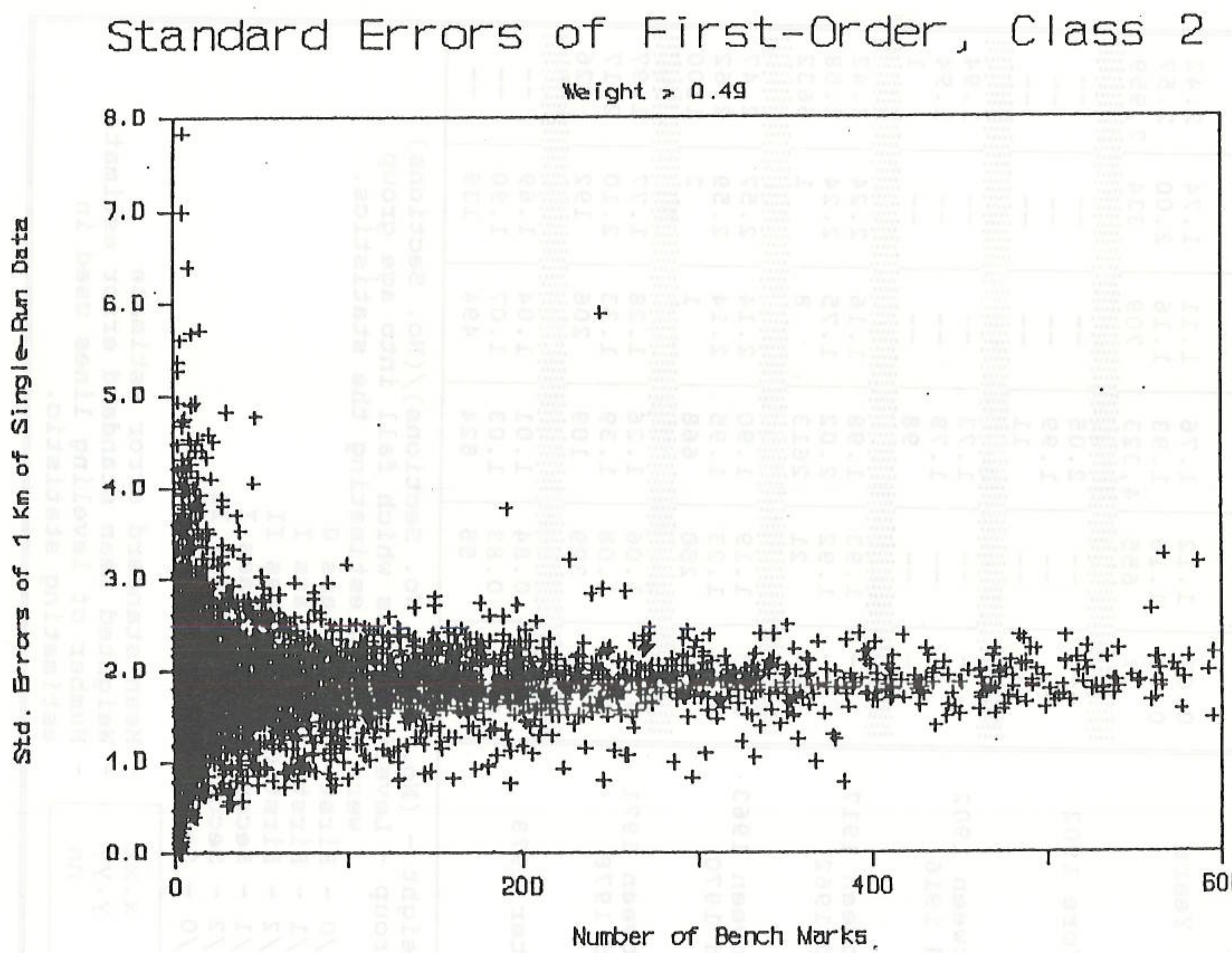

Figure 2. Plot of first-order, class II standard errors versus number of bench marks on leveling line

Table 2. Estimates of standard errors of leveling data in NGSIDB; only lines with at least 50 bench marks and at least 50 percent double-run were used to compute the statistics.

Group Lines with at least 50 BMs and at least 50% double-run	Order/Class 1/0	1/1	1/2	2/1	2/2	2/0
All Years	0.91 0.92 20	1.12 1.14 367	1.88 1.90 1,700	1.57 1.62 77	2.24 2.29 33	2.57 2.56 405
Before 1902	-- -- --	-- -- --	2.05 1.99 11	-- -- --	-- -- --	-- -- --
Between 1902 and 1916	-- -- --	-- -- --	1.68 1.69 79	-- -- --	-- -- --	1.94 1.94 1
Between 1917 and 1962	-- -- --	1.82 1.83 13	1.96 1.98 1,292	1.43 1.43 1	2.24 2.24 1	2.59 2.59 280
Between 1963 and 1970	1.13 1.13 2	1.16 1.18 130	1.75 1.77 231	-- -- --	-- -- --	2.54 2.53 92
Between 1971 and 1978	0.91 0.90 16	1.06 1.07 211	1.16 1.17 33	1.61 1.63 35	2.34 2.40 18	2.55 2.51 32
After 1978	0.73 0.73 2	0.92 0.93 13	1.06 1.06 54	1.54 1.61 41	2.10 2.15 14	-- -- --

Weight - (No. Runs - No. Sections)/(No. Sections)
Group - Leveling lines which fall into age group were used in estimating the statistics.

1/0 - First-Order, Class 0
1/1 - First-Order, Class I
1/2 - First-Order, Class II
2/1 - Second-Order, Class I
2/2 - Second-Order, Class II
2/0 - Second-Order, Class 0

x.xx - Mean standard error estimate
y.yy - Weighted mean standard error estimate
nn - Number of leveling lines used in estimating statistic.

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